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MARCH 1971 J. AIRCRAFT VOL. 8, NO. 3

Minimum Time Turns for a Supersonic Airplane at Constant Altitude

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Optimal control theory is used to determine thrust, bank-angle, and angle-of-attack programs for minimum time turns of a supersonic airplane at constant altitude for three different terminal conditions: 1) both heading angle and velocity specified, 2) only heading angle specified, 3) only velocity specified. The angle-of-attack is constrained to be less than the stall angle of the aircraft and the thrust is constrained to be less than the maximum attainable thrust, which at constant altitude is a function of velocity. Numerical results are given for a typical supersonic airplane at two different altitudes. These results show that in general a variable velocity, variable bank angle turn at full throttle is minimizing.

Nomenclature

C_{D_0} = zero lift drag coefficient
$C_{L\alpha}$ = lift coefficient curve slope ($dC_L/d\alpha$)
D = drag
D_L = drag due to lift
D_0 = zero lift drag
g = acceleration of gravity
H = Hamiltonian
L = lift
L_α = lift curve slope ($dL/d\alpha$)
m = mass of aircraft
M = Mach number
q = dynamic pressure
S = area
T = thrust
T_M = maximum thrust
V = velocity
W = weight
α = angle-of-attack
α_s = angle-of-attack at stall
β = heading angle
η = aerodynamic efficiency factor
σ = bank angle
λ_v = velocity adjoint
λ_β = heading angle adjoint
μ_1 = thrust constraint adjoint
μ_2 = angle-of-attack constraint adjoint
ρ = density of atmosphere

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Subscripts

$(\ )_0$ = initial value
$(\ )_f$ = final value

I. Introduction

OPTIMAL control theory has been used to determine optimum flight paths in a vertical plane for powered aircraft[1–4] and to determine optimum three-dimensional flight paths for unpowered lifting vehicles in the atmosphere.[5] However, very little work seems to have been done on optimum three-dimensional flight paths for powered aircraft. As a small step in this direction, Refs. 6 and 7 discussed minimum-fuel flight paths in a horizontal plane for a powered supersonic aircraft. This paper presents minimum-time flight paths in a horizontal plane for powered supersonic aircraft. The aircraft is treated as a point mass with three control variables, thrust T, bank angle σ, and angle-of-attack α. The constraint of constant altitude is used to determine α, given T, σ, and the aircraft velocity V.

II. Equations of Motion

The equations of motion, assuming a coordinated turn at constant altitude, and neglecting the change in mass because of the burning of fuel, are (see Fig. 1 for nomenclature)

$$m\dot{V} = T\cos\alpha - D \tag{1}$$

$$mV\dot{\beta} = (L + T\sin\alpha)\sin\sigma \tag{2}$$

$$mg = (L + T\sin\alpha)\cos\sigma \tag{3}$$

where $L = C_{L\alpha}\alpha qS$, $q = \frac{1}{2}\rho V^2$, $D = C_{D0}qS + \eta C_{L\alpha}\alpha^2 qS$.

Assuming $\cos\alpha \cong 1$, $\sin\alpha \cong \alpha$, and that $T\sin\alpha \ll L$,‡ Eqs. (1–3) can be rewritten as

$$m\dot{V} = T - D_0 - D_L\sec^2\sigma \tag{4}$$

$$\dot{\beta} = g\tan\sigma/V \tag{5}$$

$$\alpha = W\sec\sigma/L_\alpha \tag{6}$$

where $D_0 = C_{D0}qS$, $D_L = \eta W^2/L_\alpha$, $L_\alpha = C_{L\alpha}qS$.

III. Constant Velocity, Constant Bank Angle Turns

For constant velocity V and constant bank angle σ, it follows from Eq. (4) that

$$T = D_0(V) + D_L(V)\sec^2\sigma \tag{7}$$

If velocity V and thrust T are specified, Eq. (7) determines the required constant σ;

$$\tan\sigma = [(T - D_0 - D_L)/D_L]^{1/2} \tag{8}$$

where $\sec^2\sigma = 1 + \tan^2\sigma$ was used. The β is then given by Eq. (5) and the angle-of-attack is given by Eq. (6). It is apparent that $\dot{\beta}$ increases with T; thus, the largest $\dot{\beta}$ for any velocity will be obtained by using T_M, which is a function of velocity.

Figure 2 shows $\dot{\beta}$ vs Mach number (Mach number = V/a where a = speed of sound) at 25,000 ft and 50,000 ft for a typical supersonic airplane using maximum thrust. The thrust and aerodynamic characteristics of the airplane are given in Appendix A. It is apparent that at each altitude there is a velocity (Mach number) where $\dot{\beta}$ is a maximum. There are also several local maxima that are lower than the true maximum. Note that the required α below a certain Mach number exceeds the stall angle of the airplane (α_S = stall angle, was arbitrarily chosen to be 12° for this airplane). Figure 3 shows the corresponding values of σ vs Mach number.

‡ Equation (3) can be written for small α as $W = (L_\alpha + T)\alpha\cos\sigma$. For the airplane used in this paper, T/L_α is never larger than 2% at supersonic speeds. Thus, even though neglecting T/L_α leads to a 4% error in the drag due to lift, the error is comparable to estimated inaccuracies in our other data.

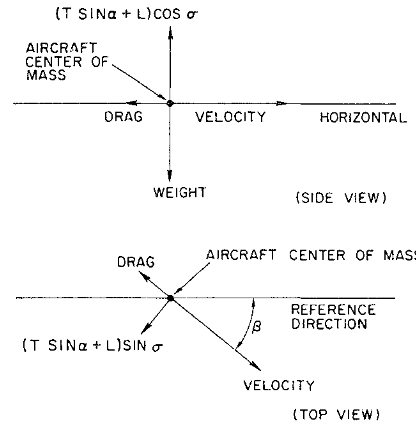

Fig. 1 Nomenclature used in analysis of supersonic aircraft.

IV. Minimum Time Turns with Final Velocity and Heading Angle Specified

A. Necessary Conditions

In general, a faster turn can be made if we do not restrict the turn to a constant velocity and bank angle. To determine the minimum time turn, we shall apply optimal control theory.[8]

First we shall determine $\sigma(t)$ and $T(t)$ to minimize t_f subject to the constraints Eqs. (4) and (5) with

$$\beta(0) = 0,\ \beta(t_f) = \beta_f \tag{9}$$

$$V(0) = V_0,\ V(t_f) = V_f \tag{10}$$

and the constraints

$$0 \leq T \leq T_M(V) \tag{11}$$

$$\alpha = W\sec\sigma/L_\alpha(V) \leq \alpha_S \tag{12}$$

where T_M = maximum thrust and α_S = stall angle.

The variational Hamiltonian for this problem may be written as

$$H = 1 + (\lambda_v/m)(T - D_0 - D_L\sec^2\sigma) + (\lambda_\beta/V)g\tan\sigma + \mu_1 T(T - T_M) + \mu_2[(W\sec\sigma/L_\alpha) - \alpha_S] \tag{13}$$

where

$$\mu_1 \begin{cases} \neq 0 & \text{if} \quad T = 0 \quad \text{or} \quad T = T_M \\ = 0 & \text{if} \quad 0 < T < T_M \end{cases} \tag{14}$$

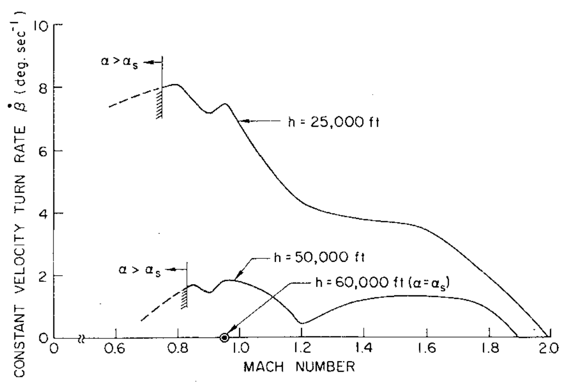

Fig. 2 Constant velocity turn rate as a function of Mach number.

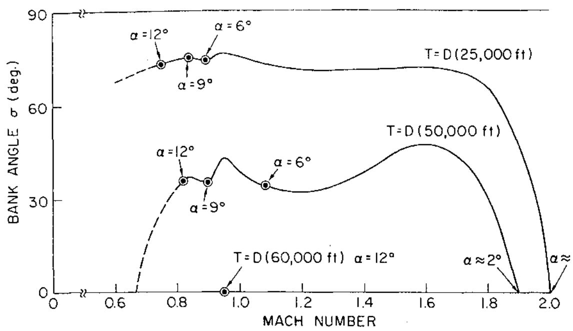

Fig. 3 Bank angle for constant velocity turns as a function of Mach number.

$$\mu_2 \begin{cases} \neq 0 \text{ if } W \sec\sigma/L_\alpha = \alpha_S \\ = 0 \text{ if } W \sec\sigma/L_\alpha < \alpha_S \end{cases} \qquad (15)$$

Necessary conditions for a minimum time turn are Eqs. (4, 5, 9–12) and

$$\dot{\lambda}_v = -\frac{\partial H}{\partial V} \equiv \frac{\lambda_v}{m}(D_0' + D_L' \sec^2\sigma) + \frac{\lambda_\beta g \tan\sigma}{V^2} + \mu_1 T T_M' + \frac{\mu_2 W \sec\sigma L_\alpha'}{L_\alpha^2} \qquad (16)$$

$$\dot{\lambda}_\beta = -\partial H/\partial\beta = 0 \qquad (17)$$

$$0 = \partial H/\partial T = (\lambda_v/M) + \mu_1(2T - T_M) \qquad (18)$$

$$0 = \frac{\partial H}{\partial \sigma} = \sec^2\sigma\left(\frac{g\lambda_\beta}{V} - \frac{2D_L\lambda_v \tan\sigma}{m} + \frac{\mu_2 W \sin\sigma}{L_\alpha}\right) \qquad (19)$$

$$\mu_1 \geq 0 \qquad (20)$$

$$\mu_2 \geq 0 \qquad (21)$$

$$H(t_f) = 0 \qquad (22)$$

Since H is not an explicit function of time, a first integral of the optimal turn is $H = $ const; from Eq. (22), this requires

$$H = 0,\ 0 \leq t \leq t_f \qquad (23)$$

Now Eqs. (14, 18, and 20) require that

$$\begin{aligned} &\lambda_v = 0 \text{ if } 0 < T < T_M \\ &\lambda_v \leq 0 \text{ if } T = T_M \text{ [since } \mu_1 = -\lambda_v/mT_M \geq 0 \text{ from Eq. (18)]} \\ &\lambda_v \geq 0 \text{ if } T = 0 \text{ [since } \mu_1 = \lambda_v/mT_M \geq 0 \text{ from Eq. (18)]} \end{aligned} \qquad (24)$$

If $0 < T < T_M$ for a finite length of time, then, Eq. (24), $\dot{\lambda}_v = 0$; Eqs. (13, 16, and 19), using $\mu_1 = 0$, it then follows that

$$1 + \frac{\lambda_\beta g \tan\sigma}{V} = 0 \qquad (25)$$

$$\frac{\lambda_\beta g \tan\sigma}{V^2} + \frac{\mu_2 W \sec\sigma L_\alpha'}{L_\alpha^2} = 0 \qquad (26)$$

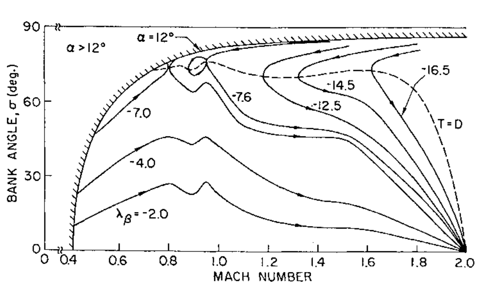

Fig. 4 Bank angle for minimum time turns at 25,000 ft as a function of Mach number.

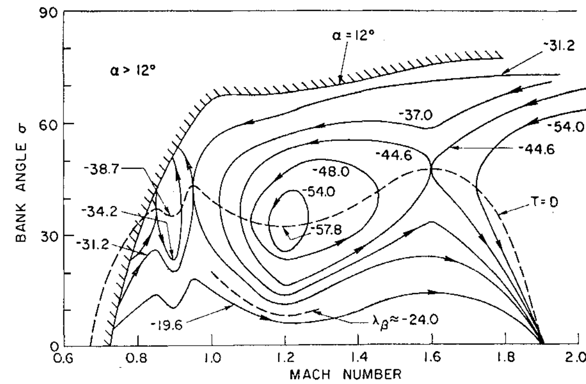

Fig. 5 Bank angle for minimum time turns at 50,000 ft as a function of Mach number.

$$\frac{g\lambda_\beta}{V} + \mu_2 \frac{W \sin\sigma}{L_\alpha} = 0 \qquad (27)$$

Clearly Eqs. (25–27) cannot be satisfied if $\mu_2 = 0$, so $\alpha = \alpha_S$ is necessary if $\lambda_v = 0$. Except for $\lambda_v = 0$, these are the conditions for a constant velocity, constant bank angle turn. Thus, Eqs. (25–27) would be satisfied only for the unlikely case where for $T < T_M$ $\dot{\beta}$ vs M has a local maximum at a point where $\alpha = \alpha_S$. At all other points intermediate values of thrust will not occur, and T will either be zero or T_M.

If $\alpha < \alpha_S$, then $\mu_2 = 0$, and Eq. (19) requires that

$$\lambda_v = (mg\lambda_\beta/2VD_L) \text{ ctn}\sigma \qquad (28)$$

It is clear that $\lambda_\beta < 0$ since $dt_f = \lambda_\beta \delta\beta$, and $\delta\beta > 0$ certainly implies $dt_f < 0$.¶ Hence, $\lambda_v < 0$ from Eq. (28), and, from Eq. (24), this requires $T = T_M$.

Substituting Eq. (28) into the first integral, Eq. (23) yields

$$\tan\sigma = -(V/g\lambda_\beta) \pm [(V/g\lambda_\beta) - (T_M - D_0 - D_L)/D_L]^{1/2} \qquad (29)$$

Substituting Eq. (29) into Eq. (4), manipulating, and using Eq. (29) again, yields

$$m\dot{V} = \pm 2D_L[(V/g\lambda_\beta)^2 - (T_M - D_0 - D_L)/D_L]^{1/2} \tan\sigma \text{ for } \alpha < \alpha_S \qquad (30)$$

where the (+) sign in Eq. (30) corresponds to the (−) sign in Eq. (29). Thus the two values of σ obtained in Eq. (29) correspond to $\dot{V} > 0$ [using (−) sign] and $\dot{V} < 0$ [using (+) sign].

Dividing Eq. (30) into Eq. (5) eliminates time;

$$d\beta/dV = \pm\, mg/2D_L V \frac{1}{[(V/g\lambda_\beta)^2 - (T_M - D_0 - D_L)/D_L]^{1/2}} \text{ for } \alpha < \alpha_S \qquad (31)$$

Since the right side of Eq. (31) is a function only of V, Eq. (31) can be integrated to give a one parameter family of optimal paths in the β, V state space (λ_β being the parameter).

B. Numerical Results

If, as is usual, the maximum thrust and aerodynamic coefficients vary significantly with Mach number, Eqs. (30) and (31) must be integrated numerically for $\lambda_\beta = $ const.§ Equation (29) gives σ directly. Our numerical results are for the airplane whose characteristics are given in Appendix A.

Figures 4 and 5 show bank angle σ vs Mach number M, for various values of λ_β at altitudes of 25,000 ft and 50,000 ft. There are three different types of unconstrained arcs: 1)

¶ From Eqs. (5, 13, and 28), λ_β can be interpreted as $\lambda_\beta = -1/(\dot{\beta})_{T=D} < 0$.

§ The simpler but less realistic case, where the variations of thrust and aerodynamic coefficients with velocity are neglected, is treated in Appendix B.

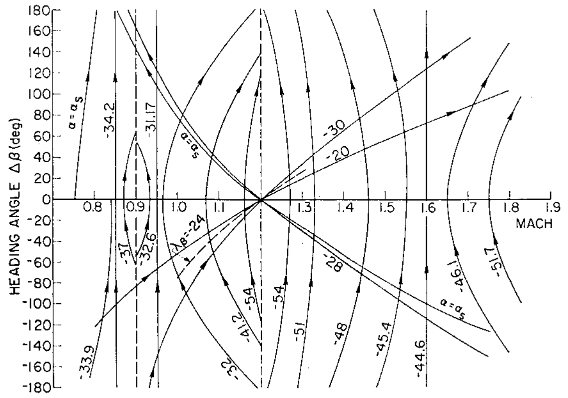

Fig. 6 Change in heading angle for minimum time turns at 50,000 ft as a function of Mach number.

purely accelerating turns at low bank angles, 2) purely decelerating turns at high bank angles, and 3) decelerating-accelerating or accelerating-decelerating turns at intermediate bank angles. As shown before, maximum thrust should be used as long as $\alpha < \alpha_S$.

In Figs. 4 and 5, at each local minimum of the $(\sigma)_{T=D}$ vs Mach number curve there is a region centered about the local minimum where the optimal paths close on themselves. In Fig. 4 this occurs at $M = 0.9$. In Fig. 5, this occurs at $M = 0.9$ and $M = 1.2$.

Necessary conditions for optimality of a constant velocity turn are Eq. (19), $T = D$, and $\dot{\lambda}_v = 0$ in Eq. (16). The points that satisfy these conditions are the local maxima and minima on the curves of Fig. 2. At 25,000 ft there are two local maxima and one local minimum, whereas at 50,000 ft there are three local maxima and two local minima. In Figs. 2 and 3, it is apparent that the local maxima and minima of the $(\sigma)_{T=D}$ vs Mach number curve and the $(\beta)_{T=D}$ vs Mach number curve for 50,000 ft occur at the same Mach numbers. Thus, it is clear in Fig. 5, that if $M(0)$ and $M(t_f)$ were chosen to be 0.85, 0.95, or 1.6, the minimum time path would be a constant velocity, constant σ turn for any β_f. If $M(0)$ and $M(t_f)$ were chosen to be 0.9 or 1.2, it is not so clear what should be done. They do satisfy our necessary conditions for a constant velocity turn, but, as seen in Fig. 2, they give the lowest local β. This problem will be treated in the next subsection.

Figure 6 shows the change in heading angle $\Delta\beta$ vs Mach number M for various values of λ_β at an altitude of 50,000 ft. The initial heading angle is arbitrary, so all of the curves may be shifted up or down. For example, suppose we wish to find the minimum time path at 50,000 ft from $M(0) = 1.0$ to $M(t_f) = 1.3$ turning through $\Delta\beta = 100°$. We look for a speed-up path on Fig. 6 that passes through both Mach numbers and has the required $\Delta\beta$; the $\lambda_\beta = -20$ path has $\Delta\beta = 20 - (-52) = 72°$, whereas the $\lambda_\beta = -30$ path has $\Delta\beta = 33 - (-104) = 137°$; interpolating, the $\lambda_\beta = -24$ path should have $\Delta\beta \cong 100°$. Referring to Fig. 5, we can interpolate the σ vs Mach number path for $\lambda_\beta = -24$; it would start with $\sigma = 20°$, decrease to about $\sigma = 8°$ at $M = 1.2$, and then increase to $\sigma = 10°$ at $M = 1.3$.

Figure 7 shows the change in time Δt vs Mach number M for various values of λ_β at an altitude of 50,000 ft. As in Fig. 6, the initial time is arbitrary, so the curves may be shifted up or down. For the example in the previous paragraph ($\lambda_\beta = -24$), it can be interpolated from Fig. 7 that the minimum time to go from $M = 1.0$ to 1.3 and turn through 100° is about 300 sec.

Figure 8 shows the information of Figs. 5 and 6 arranged conveniently for feedback control to a particular final Mach number [in this case $M(t_f) = 1.2$]. The paths of Fig. 6 are shifted up or down so that they pass through $\beta = 0$ at $M = 1.2$; from Fig. 5 it is straightforward to determine contours of constant bank angle σ on Fig. 8. Given the present Mach

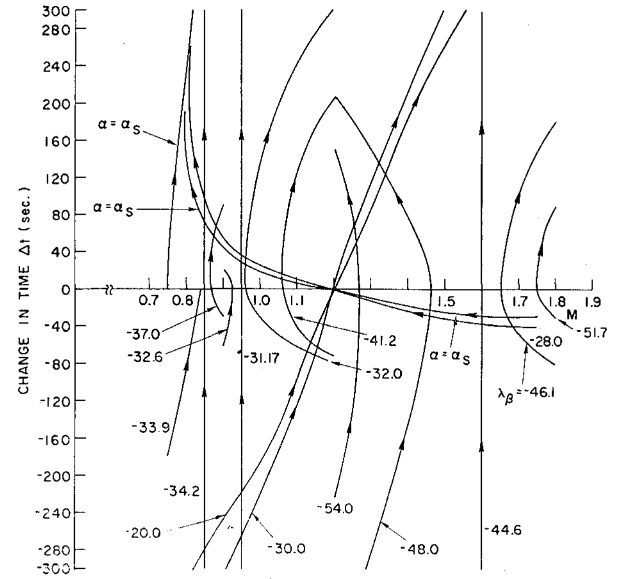

Fig. 7 Change in time for minimum time turns at 50,000 ft as a function of Mach number.

number and the heading-angle-to-go $\Delta\beta$ one can interpolate the required σ.

Figure 9 shows the information of Figs. 6 and 7 arranged conveniently to determine the minimum time-to-go to $M(t_f) = 1.2$, given the present Mach number and the heading angle-to-go.

C. Existence of Conjugate Points

In Figs. 8 and 9, if present M is 1.2 and $\Delta\beta < 100°$, the minimum time path maintains constant Mach number. However, if $\Delta\beta > 100°$, a variable Mach number path is slightly better. The point at $\Delta\beta = 100°$, $M = 1.2$ is called a "conjugate point" in the calculus of variations. Conjugate points also occur for $M_f = 0.9$ at 50,000 ft and at 25,000 ft, in fact wherever there is a local minimum in the $(\beta)_{T=D}$ vs Mach number curve.

D. A Comparison of Some Optimal and Nonoptimal Paths

From Figs. 3 and 5, it is clear that the following simple paths will give reasonably short turn times: 1) $M_f = M_0$: use $T = T_M$, constant velocity, constant bank angle; and 2) $M_f < M_0$: use $T = T_M$, $\alpha = \alpha_S$, $\sigma = \cos^{-1}W/L_\alpha\alpha_S$ until $M = M_f$, then use $T = T_M$, constant velocity, constant bank angle until $\beta = \beta_f$. 3) $M_f > M_0$: use $T = T_M$, constant velocity, constant bank angle until $\beta = \beta_f$, then use $T = T_M$,

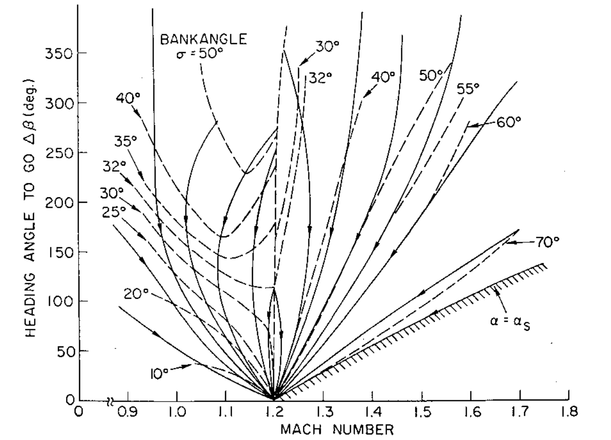

Fig. 8 Feedback control chart for $M(t_f) = 1.2$ at 50,000 ft.

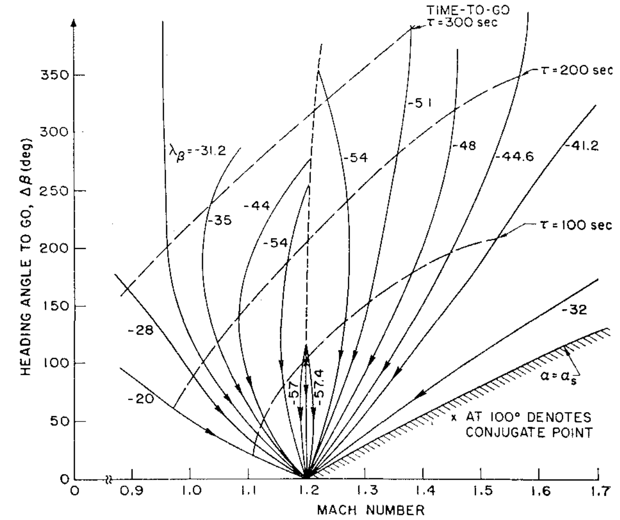

Fig. 9 Time-to-go chart for $M(t_f) = 1.2$ at 50,000 ft.

$\sigma = 0$, until $M = M_f$. Three examples using the previous strategies are given in Table 1 and the turn times compared with the minimum turn time.

V. Constant Altitude, Minimum Time Paths to a Specified Final Velocity

For these problems, if the final velocity is greater than the initial velocity, it is intuitively obvious that $T = T_M$, $\sigma = 0$ is optimum, i.e., one should use maximum thrust and the least drag possible while keeping lift equal to weight. If the final velocity is less than the initial velocity, one should use $T = 0$ and maximum drag while keeping lift equal to weight; this occurs with $\alpha = \alpha_S$ and the bank angle such that $L \cos\sigma = W$. The necessary conditions Eqs. (16–22) are satisfied by these two solutions, using the fact that $\lambda_\beta = 0$ (final β not specified).

VI. Minimum Time Turns with Only Final Heading Angle Specified

For these problems, final velocity is not specified, so $\lambda_v(t_f) = 0$. Using this fact, Eqs. (13) and (19) become

$$\left.\begin{aligned} 0 &= 1 + \lambda_\beta g \tan\sigma/V \\ 0 &= (g\lambda_\beta/V) + \mu_2 W \sin\sigma/L_\alpha \end{aligned}\right\} \text{at } t = t_f \qquad \begin{aligned}(32)\\(33)\end{aligned}$$

Clearly Eqs. (32) and (33) cannot be satisfied by $\mu_2 = 0$; hence, from Eq. (15), we have $\alpha(t_f) = \alpha_S$, i.e., the minimum time path always ends on the stall constraint boundary.

From Eqs. (5) and (6), it follows that

$$\lambda_\beta = -1/\dot{\beta}(t_f) \tag{34}$$

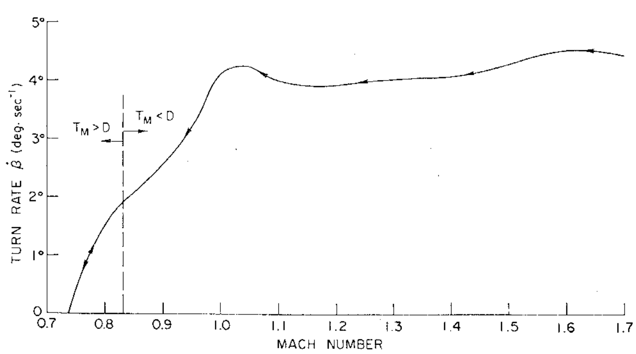

Fig. 10 Turn rate at $\alpha = \alpha_S$ at 50,000 ft as a function of Mach number.

Table 1 Comparison of optimal and nonoptimal turns

M_f	M_0	$\beta_f - \beta_0$ deg	$t_f - t_0$ sec	min $t_f - t_0$ sec
0.90	0.90	90	61	58
1.25	1.40	165	128	100
1.40	0.85	140	523	461

Figure 10 shows $\dot{\beta}$ vs Mach number for $\alpha = \alpha_S$ for the example airplane at an altitude of 50,000 ft.

From Fig. 5, it is clear that at 50,000 ft there are three possible types of paths that can satisfy the necessary conditions: 1) purely decelerating paths that intersect the constraint boundary, 2) paths that are always on the constraint boundary, and 3) accelerating-decelerating curves that intersect the $\alpha = \alpha_S$ constraint boundary. For $\alpha < \alpha_S$, $T = T_M$ but for $\alpha = \alpha_S$, T can either be zero or T_M.

From the first integral ($H = 0$), we have, using $\lambda_\beta = -1/\dot{\beta}_f$,

$$\lambda_v = [m/\dot{\beta}_f(D - T)](\dot{\beta}_f - \dot{\beta}) \tag{35}$$

Since $\lambda_v > 0$ corresponds to $T = 0$ and $\lambda_v < 0$ corresponds to $T = T_M$, it follows that, on $\alpha = \alpha_S$,

$$T = \begin{Bmatrix} T_M & \text{if } \dot{\beta} > \dot{\beta}_f \\ 0 & \text{if } \dot{\beta} < \dot{\beta}_f \end{Bmatrix} \quad \text{for } D > T_M \tag{36}$$

$$T = T_M \qquad \} \quad \text{for } D < T_M \tag{37}$$

Figure 11 shows β, σ and λ_v vs Mach number for three different types of optimal paths. For $\lambda_\beta = -33$ the airplane follows an accelerating-decelerating path with $\alpha < \alpha_S$, and then decelerates with $\alpha = \alpha_S$ until $\dot{\beta}_f = 1/33$. For $\lambda_\beta = -13.5$ the airplane decelerates along the stall boundary with thrust equal to zero until $\dot{\beta}_f = 1/13.5$. For $\lambda_\beta = -14.2$, the airplane decelerates along the stall boundary with maximum thrust until $\dot{\beta}_f = 1/14.2$.

VII. Conclusions

In general, minimum time turns at constant altitude involve a variable bank angle, variable velocity program. However, when the specified final velocity is the same as the initial velocity, a constant velocity turn is optimal for one or more discrete initial velocities and very close to optimal for all velocities.

There are three types of minimum time turns: 1) purely accelerating turns at low bank angles; 2) purely decelerating turns at high bank angles; and 3) decelerating-accelerating, constant velocity, or accelerating-decelerating turns at intermediate bank angles.

As long as the angle-of-attack is less than the stall angle, all minimum time turns require maximum thrust. If final velocity is not specified, minimum time turns end on the stall boundary; when the minimum time path coincides with the stall boundary, the thrust will either be maximum or zero.

A significant amount of time can be saved by following the minimum time paths when the initial and final velocities

Table 2 Aerodynamic coefficients as a function of Mach number

M	$C_{L\alpha}$	C_{D_0}	η
0–0.8	3.44	0.013	0.54
0.9	3.58	0.014	0.75
1.0	4.44	0.031	0.79
1.2	3.44	0.041	0.85
1.4	3.01	0.039	0.89
1.6	2.86	0.036	0.93
1.8	2.44	0.035	0.93

Table 3 Thrust as a function of Mach number

H = 25,000 ft		H = 50,000 ft	
M	$T(10^3$ lb)	M	$T(10^3$ lb)
0.2	12.8	0.2	...
0.4	13.4	0.4	4.4
0.6	14.7	0.6	4.9
0.8	16.8	0.8	5.6
1.0	19.8	1.0	6.8
1.2	23.6	1.2	8.3
1.4	28.1	1.4	10.0
1.6	32.0	1.6	11.9
1.8	34.6	1.8	13.3

differ. However, when the initial and final velocities are the same, the time saved is only slight.

Appendix A: Thrust and Aerodynamic Characteristics of a Typical Supersonic Airplane

Tables 2 and 3 give the maximum thrust at 25,000 ft and 50,000 ft and the aerodynamic coefficients as a function of Mach number. The weight of the airplane was taken as 35,000 lb and the reference area S as 530 ft^2.

Appendix B: Minimum Time Turns for the Case Where Variations in T, $C_{L\alpha}$, C_{D_0}, and η are Negligible

If variations in T, $C_{L\alpha}$, C_{D_0}, η with velocity are negligible, Eqs. (29) and (30) become

$$d\beta/dV = \pm C_1 V/(1 + C_2V^2 + C_3V^4)^{1/2} \quad \text{(B1)}$$

$$dt/dV = C_1V^2/\{g(1 + C_2V^2 + C_3V^4)^{1/2} \times [C_4V \pm (1 + C_2V^2 + C_4V^4)^{1/2}]\} \quad \text{(B2)}$$

where the C_i's are constants given by

$$C_1 = \rho S C_{L\alpha}/4\eta W,\ C_2 = (1/g^2\lambda_\beta^2) - \rho S C_{L\alpha} T/2\eta W^2$$

$$C_3 = \rho^2 S^2 C_{L\alpha} C_{D_0}/4\eta W^2,\ C_4 = -1/g\lambda_\beta$$

Equations (B1) and (B2) can be integrated in terms of simple, tabulated functions:

$$\beta = (C_1/2C_3)\log|2(C_3)^{1/2}(1 + C_2V^2 + C_3V^4)^{1/2} + 2C_3V^2 + C_2| + K_1 \quad \text{(B3)}$$

$$\frac{gt}{C_1} + K_2 = \frac{1}{2C_4(C_3)^{1/2}}\sinh^{-1}\left(\frac{2C_3V^2 + C_2}{K}\right) - \frac{1}{2C_3(A_0 - A_1)}\left[(A_0)^{1/2}\log\frac{V - (A_0)^{1/2}}{V + (A_0)^{1/2}} - A_1\log\frac{V - (A_1)^{1/2}}{V + (A_1)^{1/2}}\right] + \frac{1}{2C_4C_3(A_0 - A_1)}(G_0 - G_1 + J_0 - J_1) \quad \text{(B4)}$$

where

$$G_0 = (F_0)^{1/2} \times \log\left\{\frac{2F_0 + H_0(V^2 - A_0) - 2[F_0(C_3V^4 + C_2V^2 + 1)]^{1/2}}{V^2 - A_0}\right\}$$

$$G_1 = (F_1)^{1/2} \times \log\left\{\frac{2F_1 + H_1(V^2 - A_1) - 2[F_1(C_3V^4 + C_2V^2 + 1)]^{1/2}}{(V^2 - A_1)}\right\}$$

$$J_0 = \frac{C_2 + 2A_0C_3}{2(C_3)^{1/2}}\sinh^{-1}\left(\frac{2C_3V^2 + C_2}{K}\right)$$

$$J_1 = \frac{C_2 + 2A_1C_3}{2(C_3)^{1/2}}\sinh^{-1}\left(\frac{2C_3V^2 + C_2}{K}\right)$$

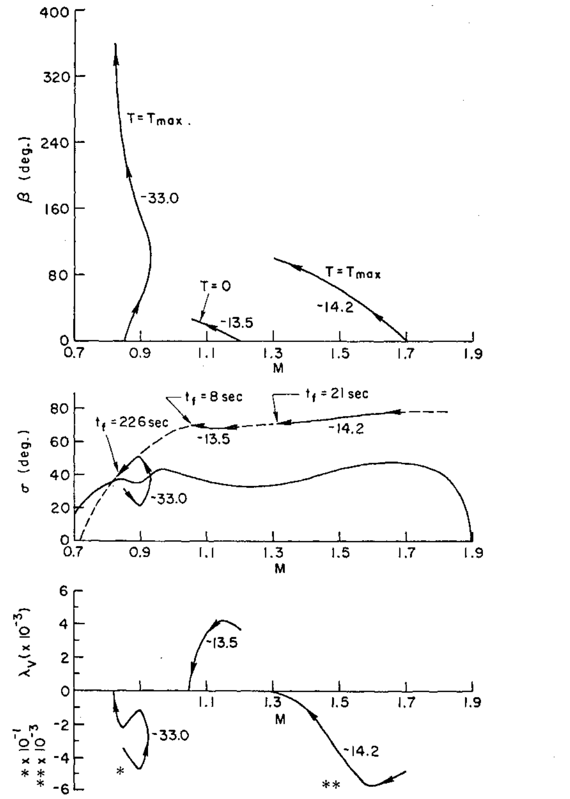

Fig. 11 Minimum time turns with final velocity unspecified at 50,000 ft.

The constants are defined as

$$K = (4C_3 - C_2^2)^{1/2}$$

$$A_0 = \frac{1}{2}\left\{\frac{C_4^2 - C_2}{C_3} + \left[\left(\frac{C_2 - C_4^2}{C_3}\right)^2 - \frac{4}{C_3}\right]^{1/2}\right\}$$

$$A_1 = \frac{1}{2}\left\{\frac{C_4^2 - C_2}{C_3} - \left[\left(\frac{C_2 - C_4^2}{C_3}\right)^2 - \frac{4}{C_3}\right]^{1/2}\right\}$$

$$F_0 = 1 + A_0C_2 + A_0^2C_3,\ F_1 = 1 + A_1C_2 + A_1^2C_3$$

$$H_0 = C_3 + 2A_0C_3,\ H_1 = C_2 + 2A_1C_3$$

It has been assumed that $4C_3 > C_2^2$, $F_1, F_2, A_1, A_2 > 0$.

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